JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 99 (1998) 119–128

On orthogonal polynomials for certain nondefinite linear functionals

Sven Ehrich

GSF Research Center for Environment and Health, Institute of Biomathematics and Biometry, Ingolstädter Landstr. 1, 85764 Neuherberg, Germany

Received 25 November 1997; received in revised form 15 April 1998

Abstract

We consider the non-definite linear functionals $L_n[f] = \int_{\mathbb{R}} w(x) f^{(n)}(x)\,dx$ and prove the nonexistence of orthogonal polynomials, with all zeros real, in several cases. The proofs are based on the connection with moment preserving spline approximation.

Keywords: Nondefinite linear functionals; Orthogonal polynomials; Nonexistence; Moment preserving spline approximation

1. Introduction and statement of the main results

For a weight function w on $\mathbb{R}$ such that $\|w\|_1 = \int_{\mathbb{R}} |w| > 0$ and all moments $\int_{\mathbb{R}} w(x)x^k\,dx$ exist, we consider orthogonal polynomials with respect to the nondefinite linear functionals

$$L_n[f] = \int_{\mathbb{R}} w(x) f^{(n)}(x)\,dx, \quad n \geqslant 1, \tag{1}$$

i.e. polynomials P_m of degree $\leqslant m$ satisfying

$$\int_{\mathbb{R}} w(x)\ (P_m(x)x^k)^{(n)}\,dx = 0, \quad k = 0, 1, \ldots, m-1. \tag{2}$$

In particular, we are interested in the problem whether there exist polynomials P_m which satisfy (2) and all of whose zeros are real. Thus, our problem is connected with the existence of real Gauss type quadrature formulas for linear functionals of the type (1). As our main results, we prove several instances of nonexistence of orthogonal polynomials P_m with all zeros real for general classes of weight functions w.

0377-0427/98/\$ – see front matter
PII: S0377-0427(98)00150-2

Suppose $w \in C^n$ (supp(w)), where supp(w) is the support of w,

$$w^{(n)} = v p_n(v), \tag{3}$$

where $v \geqslant 0$ is a weight function on $\mathbb{R}$ such that $\int_{\mathbb{R}} v \geqslant 0$ and all moments $\int_{\mathbb{R}} v(x) x^k \, dx$ exist, and $p_n(v)$ is the n-th orthonormal polynomial with respect to v. Then, by partial integration, P_m satisfies

$$\int_{\mathbb{R}} v(x) p_n(v,x) P_m(x) x^k \, dx = 0, \quad k = 0, 1, \ldots, m-1.$$

For the special case $m = n+1$, P_{n+1} is called a Stieltjes polynomial (cf. [3, 7]). The idea of regarding Stieltjes polynomials as special cases of orthogonal polynomials for nondefinite inner products goes back to Gautschi [3, p.44]. The problem whether Stieltjes polynomials, depending on v, have complex zeros, has recently attracted a lot of interest (cf. [8, 9]). In this paper, we are interested in the more general problem (2), where w is not necessarily of the type (3). We first consider the problem (2) for the Legendre weight $w = \chi_{[-1,1]}$, which serves as a prototype, and then we obtain generalisations.

Theorem 1. *Let $w = \chi_{[-1,1]}$ be the Legendre weight and $n \geqslant 1$.*

(i) *For $0 \leqslant m < \lfloor (n+1)/2 \rfloor$, every polynomial $P_m \in \mathbb{P}_m$ satisfies* (2). *For $\lfloor (n+1)/2 \rfloor \leqslant m \leqslant n$, $P_m \equiv 0$ is the unique solution of* (2).

(ii) *For $m = n+1$, $P_{n+1} \not\equiv 0$ is uniquely defined by* (2). *All zeros are real for $n \in \{0,1,2\}$ and for no other $n \in \mathbb{N}$.*

(iii) *For $n+1 < m < 2n$, every $P_m \not\equiv 0$ which satisfies* (2) *has at least one pair of complex zeros for $m > 5$. Furthermore, if m and n are odd, then* (2) *is not uniquely solvable.*

(iv) *For $m \geqslant 2n$, every polynomial of the form $P_m(x) = (1-x^2)^n Q(x)$ with $Q \in \mathbb{P}_{m-2n}$ satisfies* (2).

Here $\mathbb{P}_k$ is the space of algebraic polynomials of degree $\leqslant k$. Note that for $w = \chi_{[-1,1]}$, of course

$$\int_{\mathbb{R}} w(x) f^{(n)}(x) \, dx = f^{(n-1)}(1) - f^{(n-1)}(-1), \quad f \in C^n[-1,1],$$

and hence for $m \geqslant 2n$ the existence of real Gauss type formulas (with multiple nodes) is trivial.

On the other hand, the case $m = n+1$ is of special interest for several reasons. First, the unique solvability is guaranteed by Theorem 1, second, because it is the least number of nodes for a nontrivial discretisation of L_n, and third, because of its connection with the Stieltjes polynomials. For this case, Theorem 1 can be considerably generalised.

Theorem 2. *Let w be convex, symmetric and of bounded support in (a,b). Then P_{n+1} is uniquely defined by* (2) *and has at least one pair of complex zeros for $n > 2$.*

The Legendre weight can be considered as a multiple of the B-spline of order 1. Theorem 1 can also be generalised to w being a B-spline of higher order, and to linear combinations of such. We denote the B-spline with knots $x_1, \ldots, x_k$ by $B[x_1, \ldots, x_k]$, where each knot occurs according to its multiplicity, and we use the normalisation

$$\int_{\mathbb{R}} B[x_1, \ldots, x_k](x) \, dx = 1.$$

Theorem 3. *Let* $r \geq 1$, $w = \sum_{\mu=1}^{R} d_\mu B[y_\mu, \dots, y_{r+\mu}]$, $n \geq r$, $n < m < (r+R)(n-r+1)$. *Then* P_m *has at least one pair of complex zeros for* $m > 2r + 2R + 1$.

Being mainly interested in orthogonal polynomials P_m with real zeros, a natural question is to ask for the largest possible K such that

$$\int_{\mathbb{R}} w(x)\ (P_m(x)x^k)^{(n)}\,\mathrm{d}x = 0, \quad k = 0, 1, 2, \dots, K,$$

under the restriction that all zeros of P_m are real. The following result gives an upper bound.

Theorem 4. *Let* $r \geq 1$, $w = \sum_{\mu=1}^{R} d_\mu B[y_\mu, \dots, y_{r+\mu}]$, $n \geq r$, $n < m < (r+R)(n-r+1)$. *Suppose*

$$\int_{\mathbb{R}} w(x)\ (P_m(x)x^k)^{(n)}\,\mathrm{d}x = 0, \quad k = 0, 1, 2, \dots, K,$$

and all zeros of P_m *are real. Then necessarily* $K \leq 2r + 2R$.

Corollary 1. *Let* $r \geq 1$, $w = \sum_{\mu=1}^{R} d_\mu B[y_\mu, \dots, y_{r+\mu}]$, $n \geq r$, $n < m < (r+R)(n-r+1)$. *Suppose*

$$\int_{\mathbb{R}} w(x) f^{(n)}(x)\,\mathrm{d}x = \sum_{\nu=1}^{\tilde{m}} \sum_{\rho=0}^{\kappa_\nu - 1} a_{\nu\rho} f^{(\rho)}(x_\nu) =: Q_m[f]$$

for $f \in \mathbb{P}_{m+2r+2R+1}$, *where all* x_ν *are real and* $m = \sum_{\nu=1}^{\tilde{m}} \kappa_\nu$. *Then necessarily* $m = (r+R)(n-r+1)$ *and* $Q_m[f] = r! \sum_{\mu=1}^{R} d_\mu\, \mathrm{dvd}(y_\mu, \dots, y_{r+\mu})[f^{(n-r)}]$.

The proofs of the results are based on the connection with a certain moment preserving spline approximation problem (Lemma 1). Such problems are of independent interest and have been considered by many authors [1, 2, 4–6]. An important application are monosplines and in particular Peano kernels of quadrature formulas (cf., e.g., [5, 10]). This connection is worked out in Section 2. The proofs of the results can be found in Section 3.

2. Connection with moment preserving spline approximation

Let the divided difference (with multiple nodes)

$$\mathrm{dvd}(x_1, \dots, x_k)[f] = \sum_{\nu=1}^{\tilde{k}} \sum_{\rho=0}^{\kappa_\nu - 1} a_{\nu\rho} f^{(\rho)}(x_\nu),$$

where $k = \sum_{\nu=1}^{\tilde{k}} \kappa_\nu$, and each x_ν appears κ_ν times in $x_1, \dots, x_k$, be defined by

$$\mathrm{dvd}(x_1, \dots, x_k)[p] = \begin{cases} 0, & p \in \mathbb{P}_{k-2}, \\ 1, & p(x) = x^{k-1}. \end{cases}$$

Lemma 1. *Let $0<\|w\|_1<\infty$, $n\geqslant 1$, $n<m$. Suppose the polynomial P_m in (2) has real zeros $x_1,\dots,x_m$, and let $c_1,\dots,c_{m-n}$ be such that for all $p\in\mathbb{P}_{m-1}$*

$$\int_{\mathbb{R}} w(x)p^{(n)}(x)\,\mathrm{d}x = n!\|w\|_1\sum_{v=1}^{m-n} c_v\,\mathrm{dvd}(x_v,\dots,x_{v+n})[p].$$

Then

$$\int_{\mathbb{R}} \psi(x)x^k\,\mathrm{d}x=0,\quad k=0,1,2,\dots,2m-n-1, \tag{4}$$

where

$$\psi = w - \|w\|_1\sum_{v=1}^{m-n} c_v\,B[x_v,\dots,x_{v+n}].$$

Conversely, if $x_1,\dots,x_m$ and $c_1,\dots,c_{m-n}$ are given real numbers such that

$$\int_{\mathbb{R}} \psi(x)x^k\,\mathrm{d}x=0,\quad k=0,1,\dots,2m-n-1,$$

then $P_m(x)=c\prod_{v=1}^{m}(x-x_v)$ satisfies (2).

The conditions (4) define a moment preserving spline approximation problem. Gautschi, Milovanović et al. considered the following moment preserving spline approximation problems [1, 2, 4, 6]. For given w, find $s\in S_{m,n}$ such that

(1) $\displaystyle\int_I w(x)x^j\,\mathrm{d}x=\int_I s(x)x^j\,\mathrm{d}x,\quad j=0,1,\dots,2m+n,$

(2) $I=[0,1]$, condition (1) holds for $j=0,1,\dots,2m-1$ and

$$s^{(j)}(1)=f^{(j)}(1),\quad j=0,1,\dots,n.$$

Here I in (1) may be finite, infinite or half infinite, and $S_{m,n}$ is the space of all real spline functions of the form

$$s(x)=\sum_{j=0}^{n} a_jx^j+\sum_{k=1}^{m} b_k(x-\xi_k)_+^n,$$

with real arbitrary but fixed knots $\xi_1,\dots,\xi_m$. In [1], it is proved that if $f\in C^{n+1}[0,1]$ and $f^{(n+1)}$ never vanishes on $(0,1)$, then problems (1) (with $I=[0,1]$) and (2) have unique solutions. In [5] these results were generalised and related to the theory of monosplines.

The situation in Lemma 1 is somewhat different since the approximating spline satisfies the boundary conditions $s^{(j)}(\xi_1)=0$, $j=0,1,\dots,m-\kappa_1$, and $s^{(j)}(\xi_m)=0$, $j=0,1,\dots,m-\kappa_m$, and integration is on $\mathbb{R}$ (i.e. on $\mathrm{supp}(w)\cup\mathrm{supp}(s)$).

Finally, we note that in Lemma 1 suitable constants $c_1,\dots,c_{m-n}$ always exist uniquely.

Lemma 2. *Let $0 < \|w\|_1 < \infty$, $n \in \mathbb{N}$, and $x_1 \leqslant \cdots \leqslant x_m$ be given, $x_1 \neq x_m$. There exist unique real numbers $c_1, \ldots, c_{m-n}$ such that for every $p \in \mathbb{P}_{m-1}$*

$$\int_{\mathbb{R}} w(x) p^{(n)}(x)\,\mathrm{d}x = n!\|w\|_1 \sum_{\nu=1}^{m-n} c_\nu \operatorname{dvd}(x_\nu, \ldots, x_{\nu+n})[p].$$

3. Proofs

Proof of Lemma 1. Suppose P_m has real zeros $x_1, \ldots, x_m$ and let $c_1, \ldots, c_{m-n}$ be as in the assumption. Then for every $p \in \mathbb{P}_{2m-1}$ and suitable $q, r \in \mathbb{P}_{m-1}$

$$\begin{aligned}\int_{\mathbb{R}} w(x) p^{(n)}(x)\,\mathrm{d}x &= \int_{\mathbb{R}} w(x)[qP_m + r]^{(n)}(x)\,\mathrm{d}x = \int_{\mathbb{R}} w(x) r^{(n)}(x)\,\mathrm{d}x \\ &= n!\|w\|_1 \sum_{\nu=1}^{m-n} c_\nu \operatorname{dvd}(x_\nu, \ldots, x_{\nu+n})[r] \\ &= n!\|w\|_1 \sum_{\nu=1}^{m-n} c_\nu \operatorname{dvd}(x_\nu, \ldots, x_{\nu+n})[p],\end{aligned}$$

i.e. the linear combination of divided differences is a Gauss type formula. For every $f \in C^n(\mathbb{R})$ we have

$$n! \operatorname{dvd}(x_\nu, \ldots, x_{\nu+n})[f] = \int_{\mathbb{R}} B[x_\nu, \ldots, x_{\nu+n}] f^{(n)}(x)\,\mathrm{d}x,$$

and hence, with ψ as in the lemma, and for every $p \in \mathbb{P}_{2m-1}$,

$$\int_{\mathbb{R}} \psi(x) p^{(n)}(x)\,\mathrm{d}x = 0,$$

which is equivalent to the assertion.

Now assume that $x_1, \ldots, x_m, c_1, \ldots, c_{m-n}$ are given real numbers such that

$$\int_{\mathbb{R}} \psi(x) x^k\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, 2m-n-1.$$

Let $Q_m(x) = \prod_{\nu=1}^{m}(x - x_\nu)$, and $p \in \mathbb{P}_{2m-1}$. Then on the one hand,

$$\int_{\mathbb{R}} w(x) p^{(n)}(x)\,\mathrm{d}x = \int_{\mathbb{R}} w(x)[q(x)Q_m(x)]^{(n)}\,\mathrm{d}x + \int_{\mathbb{R}} w(x) r^{(n)}(x)\,\mathrm{d}x,$$

with $q, r \in \mathbb{P}_{m-1}$, and on the other hand

$$\begin{aligned}\int_{\mathbb{R}} w(x) p^{(n)}(x)\,\mathrm{d}x &= n!\|w\|_1 \sum_{\nu=1}^{m-n} c_\nu \operatorname{dvd}(x_\nu, \ldots, x_{\nu+n})[p] \\ &= n!\|w\|_1 \sum_{\nu=1}^{m-n} c_\nu \operatorname{dvd}(x_\nu, \ldots, x_{\nu+n})[r],\end{aligned}$$

and hence

$$\int_{\mathbb{R}} w(x)\ (Q_m(x)x^k)^{(n)}\, dx = 0, \quad k = 0, 1, 2, \dots, m-1. \qquad \square$$

Proof of Theorem 1. For $0 \leqslant m < \lfloor (n+1)/2 \rfloor$, the proof is obvious. For $\lfloor (n+1)/2 \rfloor \leqslant m \leqslant n$, suppose $P_m \not\equiv 0$ exists in (2), and let $Q(x) = P_m(x)x^{n-m}$. Then $n - m < m$ and $Q^{(n)} = C \neq 0$, which is a contradiction to (2).

The uniqueness for $m = n+1$ is easy to see since, expanding

$$P_{n+1}(x) = x^{n+1} + a_n x^n + \cdots + a_1 x + a_0,$$

the system for the a_i is triangular with no zeros on the diagonal.

For $n < m < 2n$, suppose P_m has m real zeros $x_1 \leqslant \cdots \leqslant x_m$, with the possibility of multiplicities. We cannot have $x_1 = x_2 = \cdots = x_m$ since otherwise, choosing $0 \leqslant k \leqslant m-1$ such that $m - n + k$ is even, we would have

$$\int_{-1}^{1} (x - x_1)^{m-n+k}\, dx = 0,$$

which is a contradiction. Using Lemma 1, ψ cannot be identical to 0 since $m < 2n$. Hence ψ must have at least $2m - n$ changes of sign, otherwise the polynomial which changes sign precisely with ψ would yield a contradiction to (2). Using the Budan–Fourier theorem for polynomial splines (cf. [11, Theorem 4.58]), we obtain that

$$Z_{(x_1, x_m)}\left(\mathrm{D}^+ \sum_{v=1}^{m-n} c_v B[x_v, \dots, x_{v+n}] \right) \leqslant m - n, \tag{5}$$

where $\mathrm{D}^+ f$ is the right sided derivative and $Z_{(a,b)}(s)$ counts the isolated zeros of the spline s in (a, b) with the following convention (see [11, p. 154]):

Suppose that s does not vanish identically on any interval containing t, and that $s(t-) = \mathrm{D}_- s(t) = \cdots = \mathrm{D}_-^{l-1} s(t) = 0 \neq \mathrm{D}_-^l s(t)$, while $s(t+) = \mathrm{D}_+ s(t) = \cdots = \mathrm{D}_+^{r-1} s(t) = 0 \neq \mathrm{D}_+^r s(t)$. Then we say that s has an isolated zero at t of multiplicity

$$z = \begin{cases} \alpha + 1, & \text{if } \alpha \text{ is even and } s \text{ changes sign at } t, \\ \alpha + 1, & \text{if } \alpha \text{ is odd and } s \text{ does not change sign at } t, \\ \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha = \max\{l, r\}$.

(D_- is the left sided derivative.) Thus (5) implies that there exist at most $m - n$ points $t_1, \dots, t_{m-n}$ such that $\sum_{v=1}^{m-n} c_v B[x_v, \dots, x_{v+n}]$ is monotonic in (x_1, t_1), (t_{m-n}, x_m) and each of the intervals (t_i, t_{i+1}), respectively. If $(t_i, t_{i+1}) \subset [-1, 1]$ or $(t_i, t_{i+1}) \subset (-\infty, -1) \cup (1, \infty)$, then ψ has at most one change of sign in (t_i, t_{i+1}). If $-1 \in [t_i, t_{i+1})$ $(1 \in [t_i, t_{i+1}))$ then ψ has at most three changes of sign in $[t_i, t_{i+1})$. Hence ψ has at most $m - n + 5$ changes of sign, and

$$2m - n \leqslant \#\ \text{sign changes} \leqslant m - n + 5 \ \Leftrightarrow\ m \leqslant 5.$$

For $m=n+1$, if $-1\in(t_i,t_{i+1})$ $(1\in(t_i,t_{i+1}))$ then ψ has at most two changes of sign in (t_i,t_{i+1}). Hence ψ has at most 4 changes of sign, and

$$n+2\leqslant \#\text{ sign changes}\leqslant 4 \Leftrightarrow n\leqslant 2.$$

For $n\in\{0,1,2\}$ we have, respectively,

$$P_1(x)=x,\quad P_2(x)=x^2-1,\quad P_3(x)=x^3-2x.$$

If $n<m<2n$ and m and n are odd, $m=2k+1$, consider

$$P_m(x)=x^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0.$$

The conditions (2) lead to a linear system for $a_{m-1},\dots,a_0$ which decomposes into $k+1$ equations for the k coefficients $a_{2k},\dots,a_0$ and k equations for the $k+1$ coefficients $a_{2k-1},\dots,a_1$. Hence the system cannot be uniquely solvable. Finally, for $m\geqslant 2n$ the statement is obvious. □

Proof of Theorem 2. The uniqueness follows as in the proof of Theorem 1. If the zeros of P_{n+1} are real, by the symmetry of w they lie symmetric with respect to $(a+b)/2$. For $m=n+1$ we have

$$\psi=w-\|w\|_1B[x_1,\dots,x_{n+1}].$$

Using the same argument as in the proof of Theorem (1), there exists a point t such that $B[x_1,\dots,x_{n+1}]$ is monotonically increasing in (x_1,t) and decreasing in (t,x_{n+1}). By symmetry, $t=(a+b)/2$. Since w is convex, there can be at most one change of sign of ψ in $(\max\{x_1,a\},(a+b)/2)$ and in $((a+b)/2,\min\{x_{n+1},b\})$, respectively. There are two possible changes of sign in the points a and b, and $B[x_1,\dots,x_{n+1}]$ is nonnegative outside (a,b). Hence in this case

$$n+2\leqslant \#\text{ sign changes}\leqslant 4 \Leftrightarrow n\leqslant 2. \qquad \Box$$

Proof of Theorem 3. Using Lemma 1, ψ cannot be identical to 0 since $m<(r+R)(n-r+1)$. Hence ψ must have at least $2m-n$ changes of sign and at least $r-1$ fold zeros at the boundaries. Hence $\psi^{(r-1)}$ must have at least $2m-n+r-1$ changes of sign. The function $\sum_{v=1}^{m-n}c_vB^{(r-1)}[x_v,\dots,x_{v+n}]$ is a spline of degree $n-r$ with knots $x_1,\dots,x_m$, hence by the Budan–Fourier theorem

$$Z_{(x_1,x_m)}\left(\mathrm{D}^+\sum_{v=1}^{m-n}c_vB^{(r-1)}[x_v,\dots,x_{v+n}]\right)\leqslant m-n+r-1.$$

Hence there exist at most $m-n+r-1$ points $t_1,\dots,t_{m-n+r-1}$ such that $\sum_{v=1}^{m-n}c_vB^{(r-1)}[x_v,\dots,x_{v+n}]$ is monotonic on each of the intervals (x_1,t_1), $(t_1,t_2),\dots,(t_{m-n+r-1},x_m)$, respectively. On the other hand, the function $w^{(r-1)}$ is a piecewise constant spline with $r+R$ knots (i.e. possible jumps) at $y_1,\dots,y_{r+R}$. For $r+R\leqslant m-n+r$, we count the possible changes of sign of ψ as in Theorem 1: every y_i being in an interval of monotonicity of $\sum_{v=1}^{m-n}c_vB^{(r-1)}[x_v,\dots,x_{v+n}]$ leads to at most three changes of sign of $\psi^{(r-1)}$. Suppose first that each $y_1,\dots,y_{r+R}$ lies in a different interval of monotonicity of $\sum_{v=1}^{m-n}c_vB^{(r-1)}[x_v,\dots,x_{v+n}]$. Then in the remaining intervals we have at most one change of sign of $\psi^{(r-1)}$. If there are more than one of the points y_i in an interval of monotonicity, the second and any further point can give only two changes of sign and one more remaining interval. In both cases we obtain for $r+R \leqslant m-n+r$

$$\begin{aligned} 2m-n+r-1 \leqslant \# \text{ sign changes} &\leqslant 3(r+R)+m-n+r-(r+R) \\ &= m-n+3r+2R \\ &\Leftrightarrow m \leqslant 2r+2R+1. \end{aligned}$$

For $r+R > m-n+r$, we count the possible sign changes as follows: if one of the $m-n+r-1$ points t_i lies in one of the $r+R-1$ intervals $[y_j, y_{j+1})$, then there can be at most three changes of sign in $[y_j, y_{j+1})$. Suppose first that all $t_1,\dots,t_{m-n+r+1}$ lie in $[y_1,\dots,y_{r+R}]$. In the remaining $R-m+n$ intervals $[y_j, y_{j+1})$ there can be at most two changes of sign (we count, by convention, changes of sign at the jumps with the interval right to them). Furthermore there can be one change of sign in y_{r+R}, adding up to a total of

$$3(m-n+r-1)+2(R-m+n)+1 = m-n+3r+2R-2.$$

Now for every $t_i \notin [y_1,\dots,y_{r+2}]$ there can be one additional change of sign outside $[y_1,\dots,y_{r+2}]$, but at the same time there will be one interval $[y_i, y_{i+1})$ less where three changes of sign can appear instead of two. Hence

$$\begin{aligned} 2m-n+r-1 &\leqslant \# \text{ sign changes} \\ &\leqslant m-n+3r+2R-2 \;\Leftrightarrow\; m \leqslant 2r+2R-1. \quad \square \end{aligned}$$

Proof of Theorem 4. For every $p \in \mathbb{P}_{m+k}$, $k \leqslant K$, there exist $q \in \mathbb{P}_k$, $r \in \mathbb{P}_{m-1}$ such that

$$\begin{aligned} \int_{\mathbb{R}} w(x) p^{(n)}(x)\,\mathrm{d}x &= \int_{\mathbb{R}} w(x) r^{(n)}(x)\,\mathrm{d}x \\ &= n!\|w\|_1 \sum_{v=1}^{m-n} c_v \,\mathrm{dvd}(x_v,\dots,x_{v+n})[r] \\ &= \|w\|_1 \int_{\mathbb{R}} \sum_{v=1}^{m-n} c_v B[x_v,\dots,x_{v+n}](x) p^{(n)}(x)\,\mathrm{d}x. \end{aligned}$$

From the proof of Theorem 3 we obtain for the changes of sign of the function

$$w^{(r-1)} - \|w\|_1 \sum_{v=1}^{m-n} c_v B^{(r-1)}[x_v,\dots,x_{v+n}]$$

in this case

$$m+k-n+r \leqslant \# \text{ sign changes} \leqslant m-n+3r+2R \;\Leftrightarrow\; k \leqslant 2r+2R. \quad \square$$

Proof of Lemma 2. For notational simplicity, suppose that all x_i are distinct. Let $p_v(x) = x^v$. The system

$$\sum_{k=1}^{m} b_k x_k^v = \int_{\mathbb{R}} w(x) p_v^{(n)}(x)\,\mathrm{d}x$$

has the usual interpolation matrix, which is the Vandermonde matrix in this case. Hence it is sufficient to show that every functional of the form $\mathrm{D}[f]=\sum_{k=0}^{m} b_k f(x_k)$ with $\mathrm{D}[p]=0$ for $p\in\mathbb{P}_{n-1}$ can be written in the form

$$\mathrm{D}[f]=\sum_{v=1}^{m-n} d_v \operatorname{dvd}(x_v,\ldots,x_{v+n})[f]$$

with real $d_1,\ldots,d_{m-n}$. In fact,

$$\sum_{v=1}^{m-n} d_v \operatorname{dvd}(x_v,\ldots,x_{v+n})[f]=\sum_{k=0}^{m}\tilde{b}_k f(x_k),$$

and the conditions $b_k=\tilde{b}_k$, $k=1,\ldots,m-n$ give a triangular system. For $\tilde{b}_l$, $l=m-n+1,\ldots,m$, we choose

$$f_l(x)=\prod_{\substack{\kappa=m-n+1\\ \kappa\neq l}}^{m}(x-x_\kappa),$$

hence $f_l\in\mathbb{P}_{n-1}$ and

$$\begin{aligned}\mathrm{D}[f_l]&=\sum_{k=1}^{m-n}\tilde{b}_k f_l(x_k)+\tilde{b}_l f_l(x_l)=\sum_{k=1}^{m-n} b_k f_l(x_k)+\tilde{b}_l f_l(x_l)\\&=0=\sum_{k=1}^{m-n} b_k f_l(x_k)+b_l f_l(x_l),\end{aligned}$$

hence $\tilde{b}_l=b_l$. □

References

[1] M. Frontini, W. Gautschi, G.V. Milovanović, Moment-preserving spline approximation on finite intervals, Numer. Math. 50 (1987) 503–518.

[2] W. Gautschi, Discrete approximations to spherically symmetric distributions, Numer. Math. 44 (1984) 53–60.

[3] W. Gautschi, Gauss-Kronrod quadrature – a survey, in: G.V. Milovanović (Ed.), Numerical Methods and Approximation Theory III, Nis, 1988, pp. 39–66.

[4] W. Gautschi, G.V. Milovanović, Spline approximations to spherically symmetric distributions, Numer. Math. 49 (1986) 111–121.

[5] C.A. Micchelli, Monosplines and moment preserving spline approximation, in: H. Braß, G. Hämmerlin (Eds.), Numerical Integration III, Internat. Series on Numer. Math., vol. 85, Birkhäuser, Basel, 1988, pp. 130–139.

[6] G.V. Milovanović, M.A. Kovačević, Moment-preserving spline approximation and Turán quadratures, in: R.P. Agarwal, Y.M. Chow, S.J. Wilson (Eds.), Numerical Mathematics, Proc. Conf. Singapore, Internat. Series Numer. Math., vol. 86, Birkhäuser, Basel, 1988, pp. 357–365.

[7] G. Monegato, Stieltjes polynomials and related quadrature rules, SIAM Rev. 24 (2) (1982) 137–158.

[8] S.E. Notaris, An overview of results on the existence or nonexistence and the error term of Gauss–Kronrod quadrature formulae, in: R.V.M. Zahar (Ed.), Approximation and Computation, A Festschrift in Honour of Walter Gautschi, Internat. Series Numer. Math., vol. 119, Birkhäuser, Basel, 1994, pp. 485–496.

[9] F. Peherstorfer, K. Petras, Ultraspherical Gauss–Kronrod quadrature is not possible for $\lambda > 3$, Manuscript.
[10] I.J. Schoenberg, Monosplines and quadrature formulas, in: T.N.E. Greville (Ed.), Theory and Applications of Spline Functions, Academic Press, New York, 1969, pp. 157–207.
[11] L. Schumaker, Spline Functions: Basic Theory, Wiley-Interscience, New York, 1981.